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SEIFERT FIBRED MANIFOLDS AND DEHN SURGERY

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1. INTRODUCTION

When can one obtain a Seifert fibred manifold by Dehn surgery on a knot in the 3-sphere S^3? In [1] Moser conjectured that Dehn surgery on a non-torus knot could not produce a Seifert fibred manifold, and in particular, could not produce a lens space. However certain cable knots, connected sums of two torus knots and certain hyperbolic knots turn out to be counterexamples to Moser's conjecture [2–8]. Bleiler and Litherland [9] and Wu [10] completely determined when surgeries on satellite knots produce lens spaces. On the other hand, the cyclic surgery theorem [11] and recent finite surgery theorems [12, 7, 13] gave some restrictions on the surgery slopes which produce manifolds with cyclic or finite fundamental groups. (The spherical space form conjecture states that every 3-manifold with finite fundamental group is Seifert fibred.) For instance, the cyclic surgery slopes on non-torus knots are known to be integers, and finite surgery slopes on non-torus, non-cable knots have denominators at most 2. We note that all the known examples of finite surgery slopes on such knots are integers (see [12, 7, 13]).

In this paper we shall describe those surgeries on satellite knots which give Seifert fibred manifolds (Theorems 1.2 and 1.4). As a corollary, we obtain the following result on the surgery slopes.

COROLLARY 1.1. *Let K be a satellite knot which is not cabled exactly once. If a non-trivial surgery on K yields a Seifert fibred manifold, then the surgery slope is integral. Moreover, there are at most four such surgeries; if there are four, then they are two pairs of successive integers.*

Boyer and Zhang [14] have independently obtained the first half of this result. Also see [15].

Remark. (1) The assumption "not cabled exactly once" is necessary. For example, if K is a (p, q)-cable of a torus knot, then the result of $(pqn \pm 1)/n$-surgery on K is a Seifert fibred manifold for any integer n.

(2) Combining the arguments used in this paper and a result in Boyer–Zhang [14, 15], we can improve Corollary 1.1 as follows; the number "four" in Corollary 1.1 can be replaced by "two", and if there are two then they are successive. The proof will appear in [16].

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Let us establish some terminology to state our main results.

A closed 3-manifold is *simple* if it is irreducible and does not contain an incompressible torus. For instance, lens spaces and Brieskorn homology 3-spheres are simple Seifert fibred manifolds. A Seifert fibred manifold over the 2-sphere (resp. the projective plane) is non-simple if it has at least four (resp. two) exceptional fibres.

A *satellite knot* K is a knot whose exterior contains an incompressible torus T which is not boundary-parallel. We call the solid torus V bounded by T in S^3 a *companion solid torus* of K, and a core of V a *companion knot* of K. A satellite knot K is a *(p,q)-cable knot* (or *(p,q)-cabled*) if it can be isotoped to a simple closed curve C on the boundary of a companion solid torus V of K such that C represents p(meridian) + q(longitude) in $H_1(\partial V)$. For non-triviality we assume $q \geqslant 2$. A non-satellite knot (i.e., *a simple knot*) is a torus knot or a hyperbolic knot by Thurston's uniformization theorem [17] and the torus theorem [18, 19].

We denote by $M(K;r)$ the resulting 3-manifold obtained by an r-Dehn surgery of $M(\subset S^3)$ on a knot K in M; if $M \cong S^3$, for simplicity denote by $(K;r)$ (see [20]).

Our main result is:

THEOREM 1.2 (Non-simple Seifert fibred manifolds). *Let K be a non-hyperbolic knot in S^3. If $(K;r)$ is a non-simple Seifert fibred manifold, then one of the following holds:*

(1) *K is the trefoil knot, and $r = 0$.*

(2) *K is the $(2pq \pm 1, 2)$-cable of a (p,q)-torus knot, and $r = 4pq$.*

(3) *K has a companion solid torus V whose core is a torus knot and $V - K$ admits a complete hyperbolic structure in its interior, and r is an integer. Moreover, if both $(K;m_1)$ and $(K;m_2)$ are non-simple Seifert fibred manifolds, then $|m_1 - m_2| \leqslant 1$.*

(4) *K is a connected sum of two torus knots $T_{p_1,q_1} \# T_{p_2,q_2}$, and $r = p_1q_1 + p_2q_2$.*

Hence we have:

COROLLARY 1.3. *Let K be a non-hyperbolic knot in S^3.*

(1) *If $(K;r)$ is a non-simple Seifert fibred manifold, then r is an integer.*

(2) *If both $(K;m_1)$ and $(K;m_2)$ are non-simple Seifert fibred manifolds, then $|m_1 - m_2| \leqslant 1$. Hence there are at most two surgeries on K producing non-simple Seifert fibred manifolds.*

A key to the proof of Theorem 1.2 is the operation called a *modification* of a knot. Roughly speaking, if a surgery on a satellite knot yields a non-simple Seifert fibred manifold, then a surgery on a modified knot yields a reducible 3-manifold. This observation, together with the reducible surgery theorems by Gordon–Luecke [21, 22], will readily give some estimates on the surgery slope. A modification is also used to determine the types of knots.

For simple Seifert fibred manifolds, and more generally for simple manifolds, combining recent results [23–27], we prove the following:

THEOREM 1.4. *Let K be a satellite knot in S^3 which is not cabled exactly once.*

(1) *Assume that $(K;r)$ $(r \neq \infty)$ is a simple manifold. Then r is an integer, K has a companion solid torus V whose core is a simple knot in S^3, and $K(\subset V)$ is a 1-bridge braid in the sense of Gabai [24] such that $V(K;r) \cong S^1 \times D^2$.*

(2) *If both* $(K; m_1)$ *and* $(K; m_2)$ *are simple manifolds for* $m_i \neq \infty$, *then* $|m_1 - m_2| \leqslant 1$. *Hence there are at most two non-trivial surgeries on K producing simple manifolds.*

Corollary 1.1 is a direct consequence of Corollary 1.3 and Theorem 1.4.

We obtain the following sharp result for surgeries producing Seifert homology 3-spheres.

COROLLARY 1.5 (Seifert homology 3-spheres). *Let* K *be a satellite knot in* S^3. *If a non-trivial surgery* $(K; r)$ *is a Seifert fibred homology* 3-*sphere, then* $|r| = 1$. *Moreover,* $(K; 1)$ *and* $(K; -1)$ *cannot both be Seifert fibred.*

Remark. If K is not a satellite knot, then this corollary does not hold. In fact, if K is a torus knot, then $(K; 1/n)$ is Seifert fibred for any integer n. Also if K is the figure eight knot, then both $(K; 1)$ and $(K; -1)$ are Seifert fibred.

For hyperbolic knots, all the known Dehn surgeries giving Seifert fibred manifolds are integral surgeries. All the known Seifert fibred manifolds obtained by surgery on a knot contain at most four exceptional fibres if the base spaces are S^2; and at most two if the base spaces are projective planes. We would like to pose the following conjectures.

CONJECTURES

(1) *For hyperbolic knots, only integral surgeries can yield Seifert fibred manifolds.*

(2) *If* $(K; r)$ *is a Seifert fibred manifold over the* 2-*sphere* (*resp. the projective plane*), *then it contains at most four* (*resp. two*) *exceptional fibres.*

Throughout this paper we use the symbols ∂X, $\operatorname{int} X$ and $N(X)$ to denote the boundary of X, the interior of X and a tubular neighbourhood of X, respectively.

This paper is organized as follows. In Section 2, we study Dehn surgery on simple knots producing non-simple Seifert fibred manifolds. In Section 3, we introduce *satellite diagrams* to describe satellite knots and prove Theorem 1.4. Through Sections 4–7, we study Dehn surgery on satellite knots K producing non-simple Seifert fibred manifolds $(K; r)$. In Section 4, we define (non-properly embedded) planar surfaces in the exterior of K by making use of Seifert fibrations of $(K; r)$. These planar surfaces play key roles in Sections 5 and 6. In Section 5, we consider Dehn surgery on satellite knots with a single companion. To handle this case, we define a modification of a knot. We show that if a surgery on a satellite knot K produces a non-simple Seifert fibred manifold, then a surgery on the modification of K produces a reducible manifold; we create an essential sphere in the surgered manifold from the planar surface defined in Section 4. Section 6 is devoted to the study of Dehn surgeries on satellite knots with multiple companions. The proofs of Theorem 1.2 and Corollary 1.5 are given in Section 7. Section 8 consists of the examples of graph knots producing non-simple Seifert fibred manifolds. In the final section, Section 9, we give an infinite family of non-graph knots producing non-simple Seifert fibred manifolds.

2. SURGERY ON SIMPLE KNOTS

To begin with, we determine the torus knots producing non-simple Seifert fibred manifolds by Dehn surgery. For a (p, q)-torus knot $T_{p,q}$, we assume $q > |p| \geqslant 2$. Since $T_{\pm 2,3}$ are fibred knots of genus one, $(T_{\pm 2,3}; 0)$ are Seifert fibred manifolds which are also torus bundles over S^1. Hence they are non-simple Seifert fibred manifolds.

PROPOSITION 2.1. *Let $T_{p,q}$ $(q > |p| \geqslant 2)$ be a (p,q)-torus knot. If $(T_{p,q};r)$ is a non-simple Seifert fibred manifold, then $T_{p,q}$ is the* $(2,3)$ *or* $(-2,3)$*-torus knot and* $r = 0$.

Proof. Since $q > |p| \geqslant 2$, the reducible manifold $(T_{p,q};pq) \cong L(p,q)\# L(q,p)$ cannot be Seifert fibred. So assume $r \neq pq$. Then the Seifert fibration of $S^3 - \operatorname{int} N(T_{p,q})$ extends over $(T_{p,q};r)$; the extended fibration has a base space S^2 and at most three exceptional fibres. By [28, VI.13], $(T_{p,q};r)$ contains an incompressible torus only if $r = 0$. Let T be an incompressible torus in $(T_{p,q};0)$. After an isotopy, T is vertical (i.e., a union of fibres) or horizontal (i.e., transverse to fibres) [29]. It is easy to see that T cannot be vertical and hence is horizontal. Therefore, T is a branched covering space over S^2 with three branch points $|p|$, q, $|pq|$. The Riemann–Hurwitz formula implies that $2 - (1 - 1/|p|) - (1 - 1/q) - (1 - 1/|pq|) = 0$. It follows that $(p,q) = (\pm 2, 3)$ as claimed. □

For hyperbolic knots, the following is a recent result of Gordon and Luecke [30].

PROPOSITION 2.2 (Gordon and Luecke [30]). *Let K be a hyperbolic knot in S^3.*

(1) *If $(K;m/n)$ contains an incompressible torus (e.g., $(K;m/n)$ is a non-simple Seifert fibred manifold), then $|n| \leqslant 2$.*

(2) *If $(K;m/n)$ contains a Klein bottle (e.g., $(K;m/n)$ is a Seifert fibred manifold over a non-orientable surface), then $|n| = 1$.*

3. SATELLITE DIAGRAMS

To simplify descriptions, here we introduce satellite diagrams. Let K be a satellite knot in S^3. Let $\mathscr{T}$ be the set of essential tori in $E(K) = S^3 - \operatorname{int} N(K)$ which gives the torus decomposition of $E(K)$ in the sense of Jaco–Shalen [18] and Johannson [19]. Each component of $E(K) - \bigcup \mathscr{T}$ is hyperbolic or Seifert fibred; moreover, a Seifert fibred piece is either a torus knot exterior, a cable space, or a composing space [18]. A satellite diagram, D say, for K is a tree with labelled vertices and one open edge defined as follows. Each vertex of D corresponds to a component of $E(K) - \mathscr{T}$; each edge of D corresponds to a torus in $\mathscr{T} \cup \partial E(K)$; each vertex is labelled T, Ca, Co, or H according to whether the corresponding component of $E(K) - \bigcup \mathscr{T}$ is a torus knot exterior, a cable space, a composing space, or a hyperbolic space, respectively. Note that an edge for a torus in $\mathscr{T}$ connects two vertices, but the edge for $\partial E(K)$ has one end open. For example, the satellite diagram for $T_{p_1,q_1} \# T_{p_2,q_2}$ is given in Fig. 1. For a given knot K, by the uniqueness of the torus decomposition of $E(K)$, the satellite diagram for K is uniquely determined. We shall show that if a satellite knot yields a non-simple Seifert fibred manifold by surgery, the possible satellite diagrams for such a knot are quite limited.

Let K be a satellite knot in S^3, and k a companion of K. Now we choose a tubular neighbourhood V of k containing K in its interior. Then $(K;r) = (S^3 - \operatorname{int} V) \cup V(K;r)$. Here we review what is known about the surgered manifold $V(K;r)$. In the following, the *winding number* of K in V, denoted by $\operatorname{wind}_V(K)$, is the algebraic intersection number ($\geqslant 0$) of K and a meridian disk of V.

LEMMA 3.1 (Gabai [23, 24]; Scharlemann [27]). *Suppose that $V(K;r)$ has a compressible boundary. Then we have the following:*

(1) *The winding number of K in V is not zero* ([23, *Corollary* 2.5], [27]).

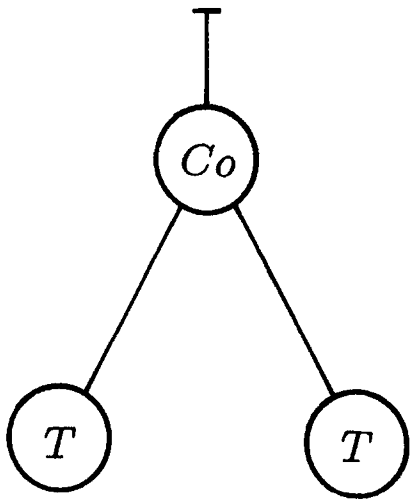


Fig. 1.

(2) *The manifold $V(K;r)$ is homeomorphic to $(S^1 \times D^2) \# M$ for some closed 3-manifold M. If $M \cong S^3$, then K is a 0 or 1-bridge braid. If $M \ncong S^3$, then $H_1(M) \ncong \{0\}$* [24].

If $(K;r)$ is a simple Seifert fibred manifold, and more generally if $(K;r)$ is a simple manifold, then by Lemma 3.1 together with [26] and [25], we prove:

THEOREM 1.4. *Let K be a satellite knot in S^3 which is not cabled exactly once. Then the following hold:*

(1) *Assume that $(K;r)$ $(r \neq \infty)$ is a simple manifold. Then r is an integer, K has a companion solid torus V whose core is a simple knot in S^3, and K is a 1-bridge braid in V such that $V(K;r) \cong S^1 \times D^2$.*

(2) *If both non-trivial surgeries $(K;m_1)$ and $(K;m_2)$ are simple manifolds, then $|m_1 - m_2| \leqslant 1$.*

Proof. Express $r = m/n$, where m and n are coprime integers. Take a companion solid torus V of K such that $S^3 - \operatorname{int} V$ is simple. Let k be a core of V. Since a simple manifold contains no incompressible torus, $V(K;m/n)$ has a compressible boundary. Then $V(K;m/n)$ is homeomorphic to $S^1 \times D^2 \# M$ for some closed 3-manifold M and by Lemma 3.1(1), $w = \operatorname{wind}_V(K) \neq 0$. Suppose that $M \ncong S^3$. Then $(K;m/n) \cong (k;m/(nw^2)) \# M$ [3]. Since the simple manifold $(K;m/n)$ is irreducible, $(k;m/(nw^2)) \cong S^3$. This contradicts Gordon–Luecke's result [26]. Therefore $M \cong S^3$ (i.e., $V(K;m/n) \cong S^1 \times D^2$); then K is a 0- or 1-bridge braid in V by Lemma 3.1(2). If K is a 0-bridge braid, K is a once cable of k, a contradiction. Hence K is a 1-bridge braid in V. The claimed results on surgery slopes follow from Lemma 3.2 in [25]. $\square$

The lemma below is our first step toward determining knots producing a non-simple Seifert fibred manifold by Dehn surgery.

LEMMA 3.2. *Let K be a satellite knot in S^3. If $(K;r)$ is a non-simple Seifert fibred manifold, then K cannot have a hyperbolic companion (i.e., a companion knot whose exterior in S^3 is a hyperbolic manifold).*

Proof. Set $r = m/n$. Suppose for a contradiction that K has a hyperbolic companion k. Let $V = N(k)$ be a companion solid torus of K; then $(K; m/n) \cong (S^3 - \operatorname{int} V) \cup V(K; m/n)$. If $V(K; m/n)$ is boundary-irreducible, then the boundary $T = \partial(V(K; m/n))$ is also incompressible in $(K; m/n)$. Since T is separating in $(K; m/n)$, it is vertical in $(K; m/n)$ with some Seifert fibration [28, VI.34]. Hence $S^3 - \operatorname{int} V$, the exterior of k, is Seifert fibred, a contradiction. Thus $V(K; m/n)$ has a compressible boundary. The argument in the proof of Theorem 1.4 implies that $V(K; m/n)$ is a solid torus and $(K; m/n) \cong (k; m/(nw^2))$, where $w = \operatorname{wind}_V(K)$. Then K is a 0- or 1-bridge braid in V (Lemma 3.1(2)). It follows that $w = \operatorname{wind}_V(K) \geqslant 2$, and hence $|nw^2| > 2$. However, applying Proposition 2.2 to $(k; m/(nw^2))$ gives $|nw^2| \leqslant 2$, a contradiction. $\square$

Using Lemma 3.2 and the argument in its proof, we can put a strong restriction on the possible satellite diagrams of satellite knots which yield non-simple Seifert fibred manifolds by surgery.

LEMMA 3.3. *Let K be a satellite knot such that $(K; m/n)$ is a non-simple Seifert fibred manifold. Assume that $(*)$ for a satellite knot with a satellite diagram listed in Fig. 2 below, only integral surgeries can produce non-simple Seifert fibred manifolds. Then the satellite diagram of K is one of those in Fig. 2.*

Proof. Let D be the satellite diagram of K. Note that an end vertex of D (i.e., a vertex with a single edge attached) corresponds to the exterior of a simple companion of K. Hence, by Lemma 3.2 none of the end vertices of D has label "H", thus they have label "T". (The vertex of label "Ca" or "Co" has more than one edges attached.) Hence the diagram D contains a subdiagram, D', which is one of three types $(H - T^s)$, $(Ca - T)$, $(Co - T^s)$ in Fig. 2.

Assume for a contradiction that $D \neq D'$. The subdiagram D' contains just one open edge, which does not correspond to $\partial E(K)$. The torus corresponding to this edge then splits S^3 into a companion solid torus of K and a 3-manifold corresponding to D'. Let V be this companion solid torus in S^3, and $k \subset S^3$ the core of V. Then the companion k of K has the satellite diagram D'. Since $(K; m/n)$, where $m/n = r$, is a non-simple Seifert fibred manifold, the same arguments in the proof of Lemma 3.2 show that $V(K; m/n) \cong S^1 \times D^2$ and

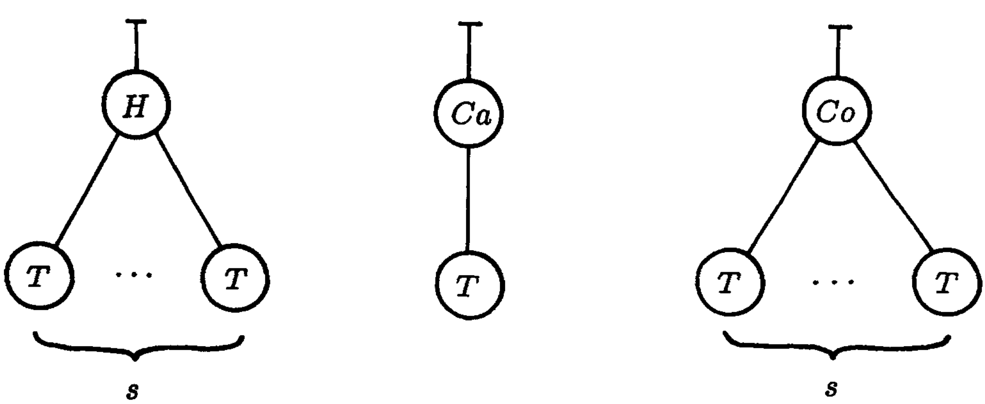


Fig. 2.

$(K; m/n) \cong (k; m/(nw^2))$, where $w = \mathrm{wind}_V(K) \geqslant 2$. By homological reasons m and nw^2 are relatively prime, so $m/(nw^2)$ is not an integer. However, applying the lemma's assumption $(*)$ to the non-simple Seifert fibred manifold $(k; m/(nw^2))$, we see that $m/(nw^2)$ is an integer, a contradiction. □

Lemma 3.3 suggests the importance of studying a satellite knot whose satellite diagram is listed in Fig. 2. In the following Sections 4 and 5, we investigate Dehn surgery on such knots, and in Section 6 we show that assumption $(*)$ in Lemma 3.2 is always the case (cf. the paragraph above Corollary 6.3).

For convenience, we use the term "a satellite knot of type ($\cdots$)" to mean "a satellite knot which has a satellite diagram of type ($\cdots$)". For instance, $T_{p_1,q_1} \# T_{p_2,q_2}$ is a satellite knot of type $(Co - T^2)$. Note also that a satellite knot of type $(Ca - T)$ is a cable of a torus knot; a satellite knot of type $(Co - T^s)$ is a connected sum of s torus knots (cf. [3]).

4. PLANAR SURFACES COMING FROM SEIFERT FIBRATIONS

Let K be a satellite knot of type $(H - T^s)$ $(s \geqslant 1)$, $(Ca - T)$, or $(Co - T^s)$ $(s \geqslant 2)$. Let $\{T_1, \ldots, T_s\}$ be the set of tori in $E(K)$ which gives the torus decomposition of $E(K)$. Then $\cup T_i$ splits $E(K)$ into the union of the piece, P, containing $\partial E(K)$ and the exteriors of torus knots, K_i, $i = 1, \ldots, s$; i.e., $E(K) = P \cup \bigcup_{i=1}^{s} E(K_i) \subset S^3$. Note that P is a hyperbolic manifold, a cable space or a composing space. Without loss of generality, $T_i = \partial E(K_i)$ for $1 \leqslant i \leqslant s$. Fig. 3 is a schematic picture of this decomposition.

For $r \in Q \cup \{\infty\}$, glue $S^1 \times D^2$ to P along $\partial E(K)$ so that the isotopy class of a meridian of $S^1 \times D^2$ corresponds to r on $\partial E(K)$ in terms of a meridian-longitude pair of K. We denote the resulting 3-manifold $P \cup_{\partial E(K)} (S^1 \times D^2)$ by $P(r)$; note $(K; r) = P(r) \cup \bigcup E(K_i)$.

LEMMA 4.1. *If $(K; r)$ is a non-simple Seifert fibred manifold, then $P(r)$ is boundary-irreducible and admits a Seifert fibration, which is the restriction of some fibration of $(K; r)$.*

Proof. We note that $S^3 = (K; \infty) = P(\infty) \cup \bigcup E(K_i)$ and that K is a core of the glued solid torus $P(\infty) - \mathrm{int}\, P$. Let W_i be the solid torus $S^3 - \mathrm{int}\, E(K_i) = P(\infty) \cup \bigcup_{j \neq i} E(K_j)$. Then $(K; r) = W_i(K; r) \cup_{T_i} E(K_i)$. We first show that $T_i = \partial(W_i(K; r))$ is incompressible in $(K; r)$. To do that it suffices to prove that $W_i(K; r)$ is boundary-irreducible. If not, $W_i(K; r) \cong (S^1 \times D^2) \# M$ for some closed 3-manifold M. Put $m/n = r$ and $w_i = \mathrm{wind}_{W_i}(K)$. It follows that $(K; m/n) \cong (K_i; m/(nw_i^2)) \# M$. By the argument in the proof of Theorem 1.4, we see that (1) $(K; m/n) \cong (K_i; m/(nw_i^2))$ and (2) $W_i(K; m/(nw_i^2)) \cong S^1 \times D^2$. Since K_i is a torus knot, assertion (1) together with Proposition 2.1 implies $m = 0$. Assertion (2) implies that K is a 0- or 1-bridge braid in W_i (Lemma 3.1(2)), hence $w_i \geqslant 2$. Since $H_1(W_i(K; 0)) \cong Z \oplus Z_{w_i}$ (see [3]), $W_i(K; 0)$ is not a solid torus, which contradicts (2). Therefore for each i, T_i is incompressible in $(K; r)$ and so in $P(r)$.

Let us show that $P(r)$ has a Seifert fibration which extends over $(K; r)$. Since T_1 is incompressible in the Seifert fibred manifold $(K; r)$, it is vertical in some Seifert fibration of $(K; r)$ ([28, VI.34]). Hence $W_1(K; r) = (K; r) - \mathrm{int}\, E(K_1)$ is Seifert fibred. Since $T_2 = \partial E(K_2)$ is an incompressible torus in the bounded Seifert fibred manifold $W_1(K; r)$, T_2 is vertical after an isotopy of the fibration of $W_1(K; r)$. Therefore we see that $(K; r) - \bigcup_{i=1}^{2} \mathrm{int}\, E(K_i)$ is Seifert fibred. Inductively, we can prove that $(K; r) - \bigcup_{i=1}^{s} \mathrm{int}\, E(K_i) = P(r)$ has a Seifert fibration extending over $(K; r)$. □

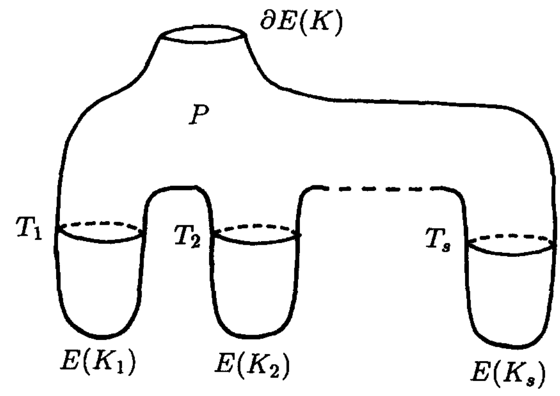


Fig. 3.

Now let us construct a planar surface in P via the Seifert fibration of $P(r)$ obtained by Lemma 4.1.

LEMMA 4.2. *There is an essential annulus $A \subset P(r)$ intersecting the glued solid torus $P(r) - \operatorname{int} P$ in at least one meridian disk; A is vertical up to isotopy with respect to the Seifert fibration of $P(r)$ obtained by Lemma* 4.1.

Proof. The proof is divided into two cases.

Case 1. K is satellite knot of type $(H - T^s)$, $s = 1$, or $(Ca - T)$.

Let $\pi: P(r) \to B$ be the Seifert fibration obtained by Lemma 4.1. Since $H_1((K;r))$ is cyclic, the base space of $(K;r)$ is the 2-sphere or the projective plane. A torus knot exterior has a unique Seifert fibration, whose base space is the 2-disk. Hence the base space B of $P(r)$ is the 2-disk or the Möbius band. Note that if B is the 2-disk, the boundary-irreducibility of $P(r)$ implies that B contains at least two exceptional points. Take a vertical annulus A in $P(r)$ as follows. If B is the 2-disk, choose A so that the arc $\pi(A)$ is essential in $B - \{\text{exceptional points}\}$. If the base space B is the Möbius band, choose A so that the arc $\pi(A)$ does not separate B.

Assume for a contradiction that A can be isotoped into P. If K is a satellite knot of type $(H - T^1)$, i.e., P is hyperbolic, then P does not contain an essential annulus. Hence A cannot be isotoped off $S^1 \times D^2 = P(r) - \operatorname{int} P$. If K is a satellite knot of type $(Ca - T)$, i.e., P is a cable space, then after an isotopy A is vertical in P. (Because A is an essential annulus in the cable space P.) This implies that P and $P(r)$ induce the same Seifert fibration on $\partial P(r)$. Then, $E(K)$ contains a Seifert fibred manifold $P \cup E(K_1)$ with incompressible boundary, absurd. Hence, A cannot be isotoped into P.

Case 2. K is a satellite knot of type $(H - T^s)$, $s \geqslant 2$, or $(Co - T^s)$.

Since $s \geqslant 2$, $P(r)$ has at least two boundary components. Hence we can take a vertical annulus $A \subset P(r)$ with $\partial A = a_1 \cup a_2$ such that $a_1 \subset T_1 = \partial E(K_1)$ and $a_2 \subset T_2 = \partial E(K_2)$; a_i is a regular fibre of $E(K_i)$. Assume for a contradiction that A can be isotoped into P.

If P is hyperbolic, then the argument in Case 1 can be applied. If P is a composing space, then without loss of generality A is vertical in P. (Because A is an essential annulus in the composing space P). Thus a component a_i of ∂A is a regular fibre of P. On the other hand, a regular fibre a_i of P cannot be a regular fibre of $E(K_i)$ for $i = 1, 2$. (Otherwise, the knot exterior $E(K)$ contains a Seifert fibred manifold $P \cup E(K_i)$!) This is a contradiction. In both

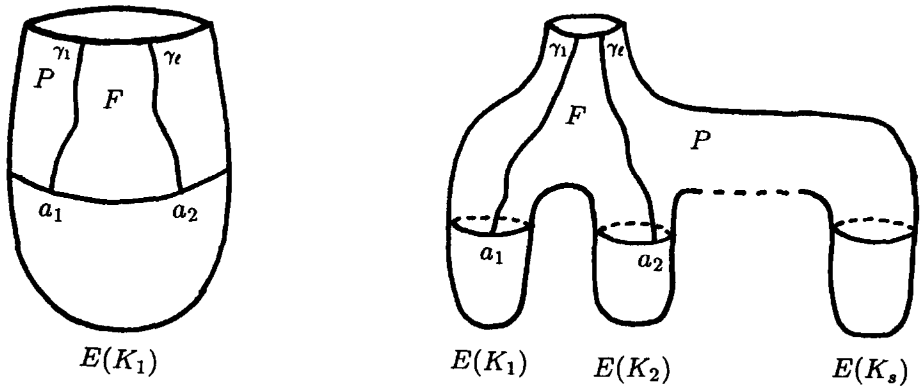


Fig. 4.

cases 1 and 2, it is straightforward to see that A can be isotoped so that A intersects $S^1 \times D^2$ in its meridian disks. □

In what follows, we assume that A is a vertical annulus by isotoping the Seifert fibration of $P(r)$. By Lemma 4.2 we can find an essential planar surface in P with certain boundary slopes. Set $F = A \cap P$, a planar surface in P. Let $a_1 \cup a_2 = \partial A$ and $\gamma_1 \cup \cdots \cup \gamma_\ell = \partial F - \partial A$; then $\partial F = a_1 \cup a_2 \cup \gamma_1 \cup \cdots \cup \gamma_\ell$. The loops γ_i are parallel loops on $\partial E(K)$ with the slope corresponding to r. If K is a satellite knot of type $(H - T^1)$ or $(Ca - T)$, then both $a_i(\subset \partial E(K_1))$ are regular fibres of $E(K_1)$. If K is a satellite knot of type $(H - T^s)$, $s \geqslant 2$, or $(Co - T^s)$, then each $a_i(\subset \partial E(K_i))$ is a regular fibre of $E(K_i)$ for $i = 1, 2$ (see Fig. 4).

5. MODIFICATION OF KNOTS AND REDUCIBLE SURGERY

Throughout this section we assume that K is a satellite knot of type $(H - T^1)$ or type $(Ca - T)$, and that $(K; r)$ is a non-simple Seifert fibred manifold. Then K has a companion solid torus V_1 whose core is a (p_1, q_1)-torus knot K_1 for some p_1, q_1; in V_1, K is a cable knot or a hyperbolic knot. Set $P = V_1 - \operatorname{int} N(K)$ as in Section 4 and $E(K_1) = S^3 - \operatorname{int} V_1$; then $E(K) = P \cup E(K_1)$. We denote by (M_1, L_1) a meridian-longitude pair of $V_1 = N(K_1)$. Paste a solid torus, say W, and V_1 along their boundaries so that the boundary of a meridian disk of W represents $L_1 + p_1 q_1 M_1 \in H_1(\partial V_1)$; then the resulting manifold is homeomorphic to a 3-sphere. We denote by K' the knot K in this new 3-sphere $V_1 \cup W$. The knot K' in $V_1 \cup W \cong S^3$ is said to be a *modification* of K (Fig. 5).

Take the slope γ (i.e., the isotopy class of an unoriented simple closed curve) on $\partial E(K) = \partial E(K')$ which corresponds to r in Q in terms of a meridian-longitude pair of K. *In what follows, we often write* $(K; \gamma)$ *for* $(K; r)$. Then $(K'; \gamma) = P(\gamma) \cup_{\partial E(K_1)} W$, where $P(\gamma) = P(r) = P \cup_{\partial E(K)} (S^1 \times D^2)$ in which $\{*\} \times \partial D^2$ has the slope γ on $\partial E(K)(\subset P)$.

Let F with $\partial F = a_1 \cup a_2 \cup \gamma_1 \cup \cdots \cup \gamma_\ell$ be the planar surface in P defined in Section 4; $\gamma_i(\subset \partial E(K))$ has the slope γ. Since each $a_i(\subset \partial E(K_i))$ is a regular fibre of $E(K_i)$, a_i represents $L_1 + p_1 q_1 M_1 \in H_1(\partial V_1)$. We can form a 2-sphere $\hat{F} \subset (K'; \gamma)$ from F by capping off $a_1 \cup a_2$ with two meridian disks of W and $\gamma_1 \cup \cdots \cup \gamma_\ell$ with ℓ meridian disks of the glued $S^1 \times D^2$. We prove that $\hat{F}$ is an essential sphere in $(K'; \gamma)$. Before that let us establish the coordinate change from a meridian-longitude pair of K to that of K'.

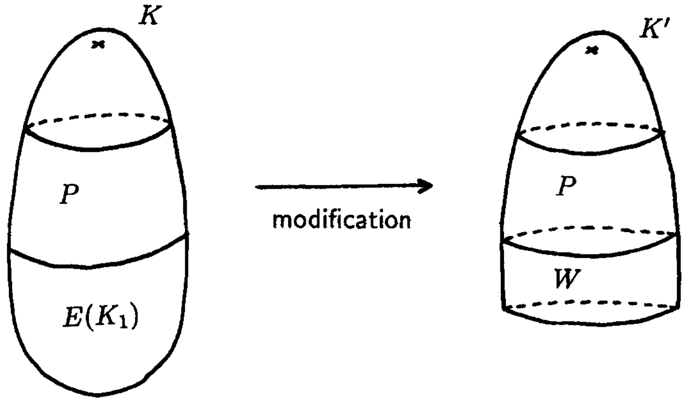


Fig. 5.

LEMMA 5.1. *Let (μ, λ), (μ', λ') denote meridian-longitude pairs of K, K', respectively; they are two bases of $H_1(\partial E(K)) = H_1(\partial E(K'))$. Then we have:*

(1) $(\mu', \lambda') = (\mu, \lambda + w^2 p_1 q_1 \mu)$, *where* $w = \mathrm{wind}_{V_1}(K)$;

(2) *A simple closed curve on $\partial E(K)$ with the slope m/n in terms of (μ, λ) corresponds to $(m - w^2 p_1 q_1 n)/n$ in terms of (μ', λ').*

Proof. Reversing the orientations of (μ', λ') and (M_1, L_1) if necessary, we have $\mu' \sim \mu$, $M_1 \sim w\mu$ and $wL_1 \sim \lambda$ in $V_1 - \mathrm{int}\, N(K)$. (Here and subsequently, "$\sim$" means "is homologous to".) When modifying K to K', $L_1 + p_1 q_1 M_1$ is null-homologous in $E(K') = (V_1 - \mathrm{int}\, N(K)) \cup W$. It follows that $\lambda + w^2 p_1 q_1 \mu \sim wL_1 + w p_1 q_1 M_1 \sim w(L_1 + p_1 q_1 M_1) \sim 0$ in $E(K')$. Hence $\lambda' = \lambda + w^2 p_1 q_1 \mu$ in $H_1(\partial E(K))$, as claimed in (1). Assertion (2) follows from (1). □

LEMMA 5.2. *The 2-sphere $\hat{F}$ is essential in $(K'; \gamma)$.*

Proof. We divide into two cases depending upon whether the base space of $P(\gamma)$ is the 2-disk or the Möbius band (see Section 4).

Case 1. The base space of $P(\gamma)$ is the 2-disk.

In this case, as observed in Section 4, $P(\gamma)(= P(r))$ has at least two exceptional fibres. Cutting $P(\gamma)$ along the vertical annulus $A = \hat{F} \cap P(\gamma)$, we obtain a Seifert fibred manifold N_i', $i = 1, 2$, over the 2-disk with at least one exceptional fibre. Hence, the 2-sphere $\hat{F}$ splits $(K'; \gamma)$ into two manifolds N_1 and N_2 such that N_i is obtained from N_i' by adding a 2-handle h_i^2 along a simple closed curve parallel to a_1. (Fig. 6).

Then, by the claim below, each N_i is a connected sum of lens spaces with a 3-ball removed. Hence, $\hat{F}$ is essential in $(K'; \gamma)$.

CLAIM 5.3. *Let M_k be a Seifert fibred manifold over the disk with k exceptional fibres. Add a 2-handle h^2 to M_k along a regular fibre of M_k. Then the resulting 3-manifold $\hat{M}_k$ is homeomorphic to a connected sum of k lens spaces minus a 3-ball, where the lens spaces do not include S^3 or $S^2 \times S^1$.*

Outline of Proof. We construct a disk separating $\hat{M}_k$ into a lens space with a puncture and some $\hat{M}_{k-1}$. Then the claim follows by induction on k. It is not difficult to find a vertical

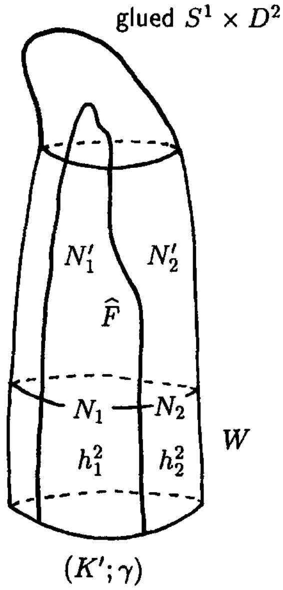


Fig. 6.

annulus A in M_k such that one component of $M_k - A$ contains just one exceptional fibre, and isotope A in M_k such that one component of ∂A, say α, lies in the attaching annulus $\partial M_k \cap \partial h^2$, and $\partial A - \alpha \subset \partial M_k - \partial h^2$. Let $D \subset \hat{M}_k$ be the union of the annulus A and the core disk of h^2 with boundary α. Then D is the desired disk. □

Case 2. The base space of $P(\gamma)$ is the Möbius band.

Since $A = \hat{F} \cap P(\gamma)$ is non-separating in $P(\gamma)$ (Section 4), $\hat{F}$ is a non-separating 2-sphere in $(K';\gamma)$ and hence essential. (This case can happen only if K' is trivial in S^3 by [31, Corollary 8.3].) This completes the proof of Lemma 5.2. □

By making use of the above modification of a knot we establish the following principle.

PRINCIPLE 5.4. *Let K be a satellite knot of type $(H - T^1)$ or $(Ca - T)$. If $(K;\gamma)$ is a non-simple Seifert fibred manifold, then $(K';\gamma)$ is a reducible manifold.*

Applying this, we obtain:

PROPOSITION 5.5. *Let K be a satellite knot of type $(H - T^1)$. If $(K;r)$ is a non-simple Seifert fibred manifold, then r is an integer. Moreover, if both $(K;m_1)$ and $(K;m_2)$ are non-simple Seifert fibred manifolds, then $|m_1 - m_2| \leqslant 1$.*

Proof. Let γ be the surgery slope corresponding to r. Then $(K';\gamma)$ is reducible by Principle 5.4. Applying [21, Theorem 1], we see that γ is an integral slope, i.e., $[\gamma] = m\mu' + \lambda'$ for some integer m in terms of a meridain-longitude pair (μ', λ') of K'. Hence, γ represents $(m + w^2 p_1 q_1)\mu + \lambda$ by Lemma 5.1, so that $r = m + w^2 p_1 q_1$, an integer. This proves the first assertion.

Suppose that $(K;m_1)$ and $(K;m_2)$ are non-simple Seifert fibred manifolds. Then by Principle 5.4 and Lemma 5.1 both $(K';m_1 - w^2 p_1 q_1)$ and $(K';m_2 - w^2 p_1 q_1)$ are reducible.

From [22] ([32, Theorem 3.3]), it follows that $|m_1 - m_2| = |(m_1 - w^2p_1q_1) - (m_2 - w^2p_1q_1)| \leqslant 1$. □

Proposition 5.5 holds also for a satellite knot of type $(Ca - T)$. However, for such a knot we obtain the definitive result as follows.

PROPOSITION 5.6. *Let K be a (p,q)-cable of a (p_1,q_1)-torus knot K_1 $(q \geqslant 2)$.*

(1) *If $q \geqslant 3$, then $(K;r)$ cannot be a non-simple Seifert fibred manifold for any r.*

(2) *If $q = 2$ and $(K;r)$ is a non-simple Seifert fibred manifold, then $p = 2p_1q_1 \pm 1$ and $r = 4p_1q_1$.*

Proof. Since P is a cable space, $P(r)$ is one of the following: $S^1 \times D^2$, $(S^1 \times D^2)\#$ (lens space) and a Seifert fibred manifold over the 2-disk with two exceptional fibres one of which has index $q \geqslant 2$ (see [3]). Lemma 4.1 implies that the first two cases cannot occur. Hence $P(r)$ has the last form; the Seifert fibration of P extends over $P(r)$. If $q \geqslant 3$, then $P(r)$ has an exceptional fibre of index $q(\geqslant 3)$. It follows that a Seifert fibration of $P(r)$ is unique up to isotopy [28, Theorem VI.18]. On the other hand, if $(K;r)$ is a non-simple Seifert fibred manifold, then it restricts to a fibration on $P(r)$ by Lemma 4.1. The Seifert fibration of $P(r)$ is, by its uniqueness, isotopic to the extension of the fibration of P. This in turn implies that $E(K) = P \cup E(K_1)$ is also Seifert fibred, a contradiction. Hence, $(K;r)$ cannot be a non-simple Seifert fibred manifold if $q \geqslant 3$.

Next we assume that $(K;r)$ is a non-simple Seifert fibred manifold for the $(p,2)$-cable of a (p_1,q_1)-torus knot K. Let γ be the slope on $\partial N(K)$ corresponding to r, and K' the knot obtained by modifying K as in the first paragraph of this section. Since K is the $(p,2)$-cable of a (p_1,q_1)-torus knot, K' is a $(p - 2p_1q_1, 2)$-torus knot in the (new) 3-sphere. Now $(K';\gamma)$ is reducible by Principle 5.4. If $|p - 2p_1q_1| \neq 1$, then K' is non-trivial, and thus the reducible surgery slope γ represents $2(p - 2p_1q_1)\mu' + \lambda'$. This slope represents $2p\mu + \lambda$ by Lemma 5.1. It follows $r = 2p$. Then $P(r) \cong S^1 \times D^2 \# L(2,p)$ [3], contradicting Lemma 4.1. Therefore $|p - 2p_1q_1| = 1$; K' is a trivial knot in S^3. The reducible surgery slope γ, then, is a longitude λ' of K'. By Lemma 5.1 $\lambda' = (4p_1q_1)\mu + \lambda$, hence $r = 4p_1q_1$. This establishes (2). □

The manifold obtained by the $4pq$-surgery along the $(2pq \pm 1, 2)$-cable of a (p,q)-torus knot is referred to as a graph manifold in [3]. In Section 8 we shall show that it is actually a non-simple Seifert fibred manifold.

6. SURGERY ON SATELLITE KNOTS WITH MULTIPLE COMPANIONS

In this section we deal with satellite knots of type $(H - T^s)$ $(s \geqslant 2)$ and type $(Co - T^s)$. In Proposition 6.8 we prove that no satellite knots of type $(H - T^s)$, $s \geqslant 2$, produce non-simple Seifert fibred manifolds by surgery. We shall also determine satellite knots of type $(Co - T^s)$ producing Seifert fibred manifolds by surgery (Proposition 6.6).

Let K be a satellite knot of type $(H - T^s)$, $s \geqslant 2$, or type $(Co - T^s)$. As shown in Section 4, the exterior of K can be expressed as $P \cup E(K_1) \cup \cdots \cup E(K_s)$; K_i is a (p_i, q_i)-torus knot T_{p_i,q_i}, and P is a hyperbolic manifold or a composing space (Fig. 3).

Let $V_i = S^3 - \operatorname{int} E(K_i)$, a solid torus containing K in its interior. So each torus knot K_i is a companion of K. We denote a meridian-longitude pair of V_i by (M_i, L_i).

LEMMA 6.1. *Each solid torus V_i, $1 \leqslant i \leqslant s$, contains a meridian disk which is disjoint from $\bigcup_{j \neq i} E(K_j)(\subset V_i)$. Consequently, each boundary component of $\bigcap_{i=1}^{s} V_i$ is compressible in $\bigcap_{i=1}^{s} V_i$.*

Proof. We only prove V_1 contains a meridian disk disjoint from $\bigcup_{j \neq 1} E(K_j)$. Let D be a meridian disk of V_1 such that $|D \cap \bigcup_{j \neq 1} \partial E(K_j)|$ is minimum among all meridian disks of V_1. Assume for a contradiction $D \cap \bigcup_{j \neq 1} E(K_j) \neq \emptyset$. Let B be the closure of an innermost disk component of $D - \bigcup_{j \neq 1} \partial E(K_j)$. Then $\partial B \subset \partial E(K_{i_0})$ for some i_0, and either $B \subset E(K_{i_0})$ or $B \subset V_{i_0}$ holds. If the former case holds, we can isotope B and thus D off $E(K_{i_0})$ using the boundary-irreducibility and the irreducibility of $E(K_{i_0})$. This isotopy reduces $|D \cap \bigcup_{j \neq 1} \partial E(K_j)|$, a contradiction. It follows that $B \subset V_{i_0}$. By the minimality assumption ∂B is essential in ∂V_{i_0}, so that B is a meridian disk of V_{i_0}. Choosing a sufficiently small tubular neighbourhood of B, we have a 3-ball $N(B) \cup E(K_{i_0})$ in V_1 which is disjoint from $E(K_j)$ for all $j \neq i_0$. Thus we can isotope D off the 3-ball and so $E(K_{i_0})$, which contradicts the minimality assumption on D. □

Assume that $(K;r)$ is a non-simple Seifert fibred manifold. We denote by γ the slope on $\partial E(K)$ corresponding to $r \in Q$ in terms of a meridian-longitude pair of K. Recall that P contains a planar surface F with $\partial F = a_1 \cup a_2 \cup \gamma_1 \cup \cdots \cup \gamma_\ell$ (Section 4); $\partial F \cap \partial E(K_i) = a_i$ $(i = 1, 2)$, a regular fibre of $E(K_i)$, and $\partial F \cap \partial E(K) = \gamma_1 \cup \cdots \cup \gamma_\ell$, parallel loops with the slope γ.

LEMMA 6.2. *Let K be a satellite knot of type $(H - T^s)$, $s \geqslant 2$, or $(Co - T^s)$. If $(K;r)$ is a non-simple Seifert fibred manifold, then r is an integer and* $\mathrm{wind}_{V_i}(K) = 1$ *for* $1 \leqslant i \leqslant s$.

This lemma together with Propositions 5.5 and 5.6 shows that a satellite knot of type $(H - T^s)$, $(Ca - T)$ or $(Co - T^s)$ can produce a non-simple Seifert fibred manifold only by integral surgery. Thus, Lemma 3.3 implies the following restriction.

COROLLARY 6.3. *If a satellite knot produces a non-simple Seifert fibred manifold by surgery, then its satellite diagram is of either type $(H - T^s)$, $(Ca - T)$, or $(Co - T^s)$, and the surgery slope is integral.*

Proof of Lemma 6.2. We prove that r is an integer and $\mathrm{wind}_{V_1}(K) = 1$. (Its proof with a slight shift implies $\mathrm{wind}_{V_i}(K) = 1$ for other i.) We modify K as in Section 5. Glue a solid torus W to $V_2 = S^3 - \mathrm{int}\, E(K_2)$ so that the loop a_2 bounds a meridian disk of W. The result $M = V_2 \cup W$ is homeomorphic to a 3-sphere because $[a_2] = L_2 + p_2 q_2 M_2 \in H_1(\partial V_2)$. We denote by K'_2 the knot K in the 3-sphere M; regard $N(K) = N(K'_2)$. After this modification of K, let V'_1 be the solid torus $M - \mathrm{int}\, E(K_1)$ containing K'_2 in its interior; $V'_1 = (V_1 - \mathrm{int}\, E(K_2)) \cup W$; see Fig. 7. (To prove $\mathrm{wind}_{V_i}(K) = 1$ in general, take some $j \neq i$ and modify K to a knot K'_j in a 3-sphere $M = V_j \cup W$ defined similarity to K'_2. Then replace K'_2 and V'_1 by K'_j and $V'_i = M - \mathrm{int}\, E(K_i)$ in the discussion below.)

By Lemma 6.1, V_1 contains a meridian disk also contained in V'_1; thus it is a meridian disk of V'_1. It follows that $\mathrm{wind}_{V_1}(K) = \mathrm{wind}_{V'_1}(K'_2)$, and that (M_1, L_1) is a meridian-longitude pair of both V_1 and V'_1.

Let $X = V'_1 - \mathrm{int}\, N(K'_2)$. We have a punctured disk $\hat{D} = F \cup_{a_2}$ (a meridian disk of W) in X such that $\partial \hat{D} = a_1 \cup \gamma_1 \cup \cdots \cup \gamma_\ell$. After the γ-surgery of V'_1 on K'_2, $\hat{D}$ gives rise to a compressing disk D, say, of $V'_1(K'_2;\gamma)$. Since $\partial D = a_1$ is a regular fibre of the exterior of the (p_1, q_1)-torus knot K_1, we have $p_1 q_1 M_1 + L_1 \sim \partial D \sim 0$ in $V'_1(K'_2;\gamma)$. Now let

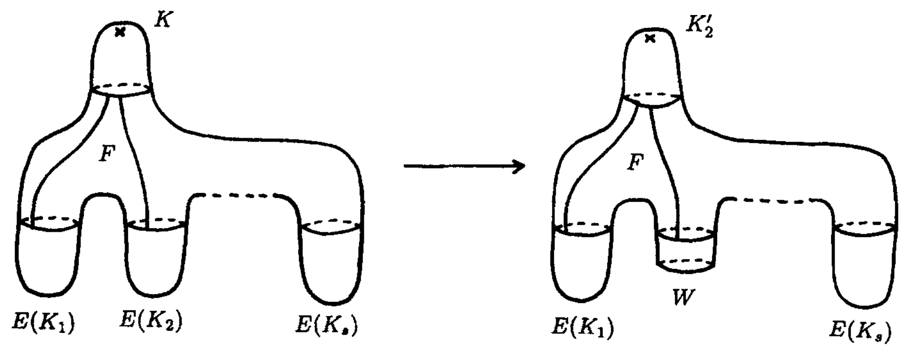


Fig. 7.

$\mu_2', \lambda_2' \subset \partial N(K_2')$ be a meridian-longitude pair of K_2', and write $[\gamma] = m'\mu_2' + n'\lambda_2' \in H_1(\partial N(K_2'))$, where m' and $n'(\geqslant 0)$ are coprime integers. The pair $\{\mu_2', L_1\}$ forms a basis of $H_1(X) \cong Z \oplus Z$. For simplicity put $w = \mathrm{wind}_{V_1'}(K_2')$. Then $M_1 \sim w\mu_2'$ and $wL_1 \sim \lambda_2'$ in X, so that

$$H_1(V_1'(K_2';\gamma)) \cong Z_{\langle \mu_2' \rangle} \oplus Z_{\langle L_1 \rangle}/(m', n'w).$$

Since $p_1q_1M_1 + L_1 = 0$ in this module, $p_1q_1M_1 + L_1 = p_1q_1w\mu_2' + L_1$ has to be a multiple of $m'\mu_2' + n'wL_1$. It follows that $n'w = 1$, so $n' = w = 1$. The equality $w = 1$ implies $\mathrm{wind}_{V_1}(K) = \mathrm{wind}_{V_1'}(K_2') = 1$, as claimed. The equality $n' = 1$ implies $[\gamma] = m'\mu_2' + \lambda_2'$. We need to express $[\gamma]$ in terms of a meridian-longitude pair (μ, λ) for K. By homological calculations, we see that $\mu_2' = \mu$ and $\lambda_2' = \lambda + p_2q_2w^2\mu = \lambda + p_2q_2\mu$. (cf. Lemma 5.1). Then $[\gamma] = (m' + p_2q_2)\mu + \lambda$, so that $r = m' + p_2q_2$, an integer. □

LEMMA 6.4. *Let K be a satellite knot of type $(H - T^s)$, $s \geqslant 2$, or $(Co - T^s)$. If $(K;r)$ is a non-simple Seifert fibred manifold, then $s = 2$, i.e., K has exactly two companions.*

Proof. In the proof of Lemma 6.2, $K(\subset S^3)$ is modified to K_2' in $V_2 \cup W \cong S^3$. Here we modify K to K' as follows. Remove the interiors of $E(K_1)$ and $E(K_2)$ from S^3, and attach solid tori W_1 and W_2 so that $a_i(\subset \partial E(K_i))$ bounds a meridian disk D_i of W_i. Call the resulting manifold (again) M. Since $V_2 \cup W_2 \cong S^3$, $V_1' = (V_1 - \mathrm{int}\, E(K_2)) \cup W_2$ is a solid torus. The proof of Lemma 6.2 shows that a meridian $M_1 = \partial V_1$ is also a meridian of V_1'. Thus, the fact $[\partial D_1] \cdot M_1 = 1$ implies $M = V_1' \cup W_1 \cong S^3$. We denote by K' the knot K in this new 3-sphere M, and say that K' is a *modification* of K (Fig. 8).

We regard $N(K) = N(K')$. Then $E(K') = M - \mathrm{int}\, N(K')$ contains a punctured sphere $S = F \cup D_1 \cup D_2$ with $\partial S = \gamma_1 \cup \cdots \cup \gamma_s \subset \partial E(K')$. In the surgered manifold $(K';\gamma)$, we can cap off the components of ∂S by meridian disks of the glued solid torus $(K';\gamma) - \mathrm{int}\, E(K')$. The resulting 2-sphere $\hat{S}$ is non-separating in $(K';\gamma)$ because $\hat{S}$ intersects a core of W_i $(i = 1, 2)$ geometrically just once. Hence, by Gabai [31, Corollary 8.3] we obtain the following.

CLAIM 6.5. *K' is a trivial knot in $M \cong S^3$ and γ is a longitude of K'. Moreover, $(K';\gamma)$ contains a non-separating 2-sphere $\hat{S}$ intersecting a core of each W_i geometrically once.*

Now let us prove $s = 2$. Assume the contrary $s \geqslant 3$. After the above modification of K, let V_3' be the solid torus $M - \mathrm{int}\, E(K_3)$, whose core is a (p_3, q_3)-torus knot K_3. Notice

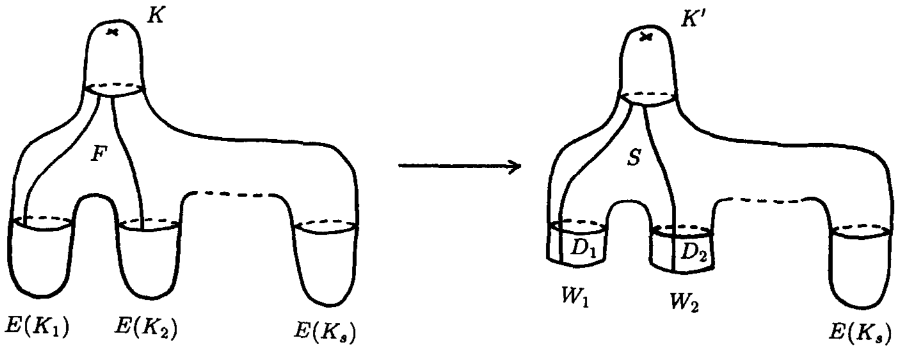


Fig. 8.

$V_3' \supset K'$. By Lemma 6.1, V_3 contains a meridian disk disjoint from $E(K_1) \cup E(K_2)$; the disk is also a meridian disk of V_3'. Hence $\mathrm{wind}_{V_3'}(K') = \mathrm{wind}_{V_3}(K)$, which is equal to 1 by Lemma 6.2. Since K' is contained in the knotted solid torus V_3' such that $\mathrm{wind}_{V_3'}(K') \neq 0$, K' is non-trivial. This contradicts Claim 6.5 above. $\square$

We now determine the knots of type $(Co - T^s)$ (i.e., the connected sum of torus knots) producing a Seifert fibred manifold. (A connected sum of unoriented knots is not uniquely determined. However, since torus knots are invertible, a connected sum of unoriented torus knots is well-defined.) Although the proposition below is proved by Kalliongis and Tsau [5], we give an alternative proof for convenience of readers.

Proposition 6.6. *Let K be the connected sum of s torus knots $T_{p_1,q_1} \# \cdots \# T_{p_s,q_s}$ $(s \geqslant 2)$. Then $(K;r)$ is Seifert fibred if and only if $K = T_{p_1,q_1} \# T_{p_2,q_2}$ (i.e., $s = 2$) and $r = p_1q_1 + p_2q_2$.*

Proof. It is well known that $(T_{p_1,q_1} \# T_{p_2,q_2};\, p_1q_1 + p_2q_2)$ is Seifert fibred (cf. [4, 5]). So assume $(K;r)$ is Seifert fibred. It is known that the result of non-trivial surgery of a composite knot contains an incompressible torus [33, 3]. $(K;r)$ is, as a result, a non-simple Seifert fibred manifold. Hence the assertion $s = 2$ follows from Lemma 6.4.

Let $K' \subset M$ $(\cong S^3)$ be the modification of K defined in the proof of Lemma 6.4; note $N(K) = N(K')$. Let (μ', λ') be a meridian-longitude pair of K'. By Claim 6.5, K' is trivial in M, and the surgery slope γ corresponding to r in terms of a meridian-longitude pair (μ, λ) of K coincides with λ'. Thus, to determine r it suffices to know the coordinate change from (μ', λ') to (μ, λ). The claim below shows $r = p_1q_1 + p_2q_2$ as desired. $\square$

Claim 6.7. $(\mu', \lambda') = (\mu, (p_1q_1 + p_2q_2)\mu + \lambda)$.

Proof. We have $M \sim \mu_i$ $(i = 1, 2)$ and $\lambda \sim L_1 + L_2$ in P. When modifying K to K', $L_1 + p_1q_1M_1$ and $L_2 + p_2q_2M_2$ are null-homologous in $E(K')$. It follows that $\lambda + (p_1q_1 + p_2q_2)\mu \sim L_1 + p_1q_1M_1 + L_2 + p_2q_2M_2 \sim 0$, and hence $\lambda' = \lambda + (p_1q_1 + p_2q_2)\mu$ in $H_1(E(K'))$. $\square$

The rest of this section is devoted to proving the following result.

Proposition 6.8. *Let K be a satellite knot of type $(H - T^s)$ with $s \geqslant 2$. Then $(K;r)$ cannot be a non-simple Seifert fibred manifold for any r.*

Proof. We know $s = 2$ by Lemma 6.4. Let a knot K' be the modification of K defined in the proof of Lemma 6.4; an ambient manifold of K' is $(S^3 - \mathrm{int}(E(K_1) \cup E(K_2))) \cup W_1 \cup W_2 \cong S^3$, where W_1, W_2 are solid tori. The ambient manifold of K' will be called S^3 not M as in the proof of Lemma 6.4. Let J_1 and J_2 be cores of W_1 and W_2, respectively. In the following, we regard $W_i = N(J_i)$. First we study the position of $J_1 \cup J_2$ in S^3.

CLAIM 6.9. *$J_1 \cup J_2$ is a trivial link in S^3.*

Proof. Each boundary component of $V_1 \cap V_2 = S^3 - \mathrm{int}(W_1 \cup W_2)$ is compressible in $V_1 \cap V_2$ by Lemma 6.1. It follow that $J_1 \cup J_2$ is trivial in S^3. □

By Claim 6.5, $(K'; \gamma)$ is homeomorphic to $S^2 \times S^1$ and contains a non-separating 2-sphere, $\hat{S}$, intersecting J_i geometrically just once for each $i = 1, 2$. The claim below states that $J_1 \cup J_2$ is in the "standard position" in $(K'; \gamma)$.

CLAIM 6.10. *$((K'; \gamma), J_1 \cup J_2)$ is homeomorphic to $(S^2 \times S^1, \{x, y\} \times S^1)$ as pairs, where $x, y \in S^2$.*

Proof. We note that $(K'; \gamma) - \mathrm{int}(W_1 \cup W_2) = (K; \gamma) - \mathrm{int}(E(K_1) \cup E(K_2)) = P(\gamma)$. By Lemma 4.1, $P(\gamma)$ admits a Seifert fibration which is a restriction of that of $(K; \gamma)$. Recall that a regular fibre $a_i (\subset \partial E(K_i))$ is a meridian of W_i. Thus $(K'; \gamma) - \mathrm{int}(W_1 \cup W_2)$ admits a Seifert fibration which restricts to a fibration by meridians on $\partial W_1 \cup \partial W_2$. Thus, the annulus $A = \hat{S} - \mathrm{int}(W_1 \cup W_2)$ is vertical. Now cutting $(K'; \gamma) \cong S^2 \times S^1$, J_1 and J_2 along the non-separating 2-sphere $\hat{S}$, we obtain $S^2 \times I$ and properly embedded arcs J_1', J_2' in $S^2 \times I$. Note that $S^2 \times I - \mathrm{int}(N(J_1') \cup N(J_2'))$ is Seifert fibred. Then attach two standard ball pairs (B_1^3, c_1) and (B_2^3, c_2) to $(S^2 \times I, J_1' \cup J_2')$ so that $J_1' \cup J_2' \cup c_1 \cup c_2$ forms a simple closed curve, k say, in $S^2 \times I \cup B_1^3 \cup B_2^3 \cong S^3$. The Seifert fibration of $S^2 \times I - \mathrm{int}(N(J_1') \cup N(J_2'))$ extends over $S^3 - \mathrm{int}\, N(k)$; the fibration restricted on $\partial N(k)$ is a fibration by meridians. This can happen only when k is a trivial knot in S^3. By the construction of k, this in turn implies that $(S^2 \times I, J_1' \cup J_2')$ is homeomorphic to $(S^2 \times I, \{x, y\} \times I)$ for some $x, y \in S^2$, thereby Claim 6.10 follows. □

Let V' be the solid torus $S^3 - \mathrm{int}\, W_1$, which contains K' and J_2. Take a slope α on $\partial N(J_2) = \partial W_2$. Since J_2 bounds a disk in V' by Claim 6.9, $V'(J_2; \alpha) \cong (S^1 \times D^2) \# L_\alpha$ where L_α denotes the lens space $(J_2; \alpha)$ (possibly S^3 or $S^2 \times S^1$); see Fig. 9(a).

We show that $V'(J_2; \alpha)$ can be naturally regarded as the result of a surgery of a solid torus on a knot different from J_2. Attach a solid torus W_2' to $(K'; \gamma) - \mathrm{int}(W_1 \cup W_2)$ along ∂W_2 so that a meridian of W_2' has the slope α on ∂W_2. Denote the resulting manifold by V_α. Since $(K'; \gamma) - \mathrm{int}(W_1 \cup W_2) \cong T^2 \times I$ by Claim 6.10, V_α is homeomorphic to a solid torus whatever the slope α is. Now let K^* be the core of the glued solid torus $(K'; \gamma) - \mathrm{int}\, E(K')$; i.e., K^* is a dual of K' after the γ-surgery. Then K^* is a knot in the solid torus V_α; $\partial N(K^*) = \partial N(K') = \partial N(K)$; see Fig. 9(b).

CLAIM 6.11. *$V_\alpha(K^*; \mu) \cong V'(J_2; \alpha)$ for any slope α, where $\mu (\subset \partial N(K^*) = \partial N(K))$ is the slope of a meridian of K.*

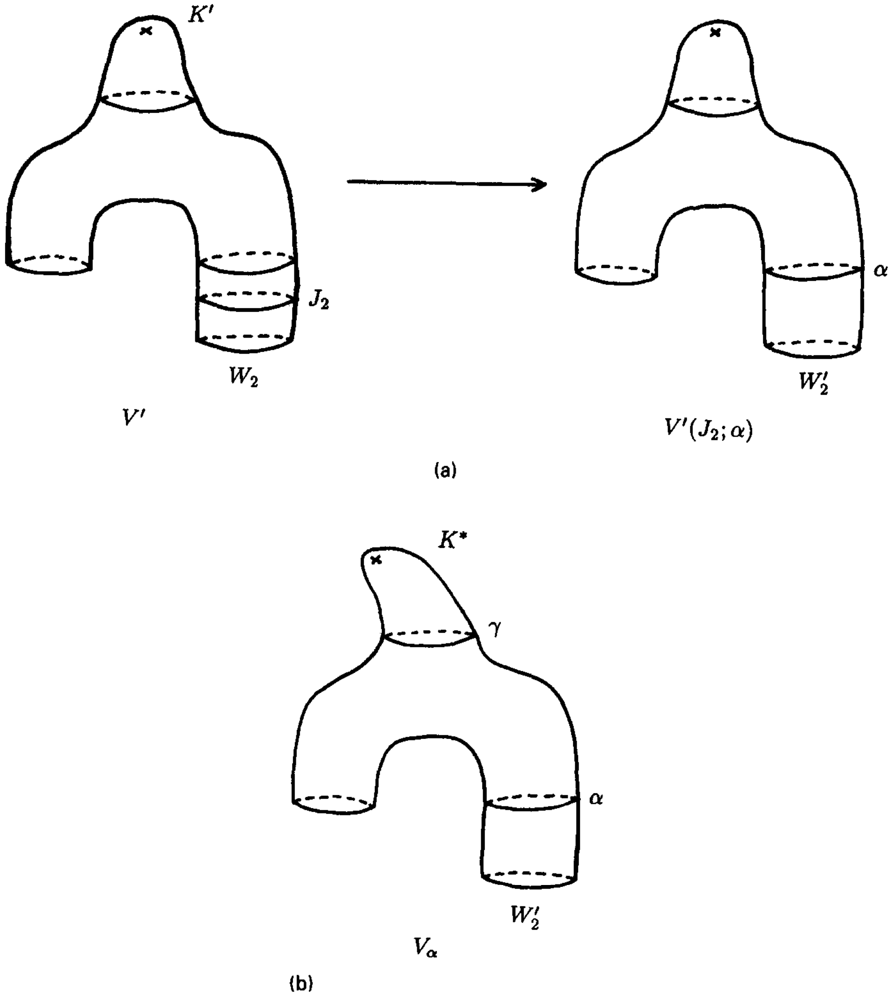


Fig. 9.

Proof. To recognize this, notice the following decompositions. For convenience, let T denote the component of ∂P such that $T = \partial N(K)$.

$$V'(J_2; \alpha) = N(K') \cup_T P \cup W_2'$$

$$V_\alpha(K^*; \mu) = N(K^*) \cup_T P \cup W_2'.$$

In these decompositions, a meridian of $N(K')$ has the slope μ on T, and that of $N(K^*)$ has the slope γ on T. Hence, removing $N(K^*)$ from V_α and sewing it back so that a meridian of the re-attached solid torus has the slope μ on T, we obtain $V'(J_2; \alpha)$. This establishes the claim. $\square$

Since $V'(J_2; \alpha) \cong (S^1 \times D^2) \# L_\alpha$, the μ-surgery of the solid torus V_α on K^* yields a reducible manifold if hopefully $L_\alpha \not\cong S^3$. Assume that $L_\alpha \not\cong S^3$ and K^* does not lie in a 3-ball in V_α; then by Scharlemann [34, Corollary 4.4] we can conclude that K^* is cabled in V_α and the cabling annulus has the slope μ. The claim below shows that the assumption above holds for infinitely many slopes α.

Let $(\mu_{J_2}, \lambda_{J_2})$ be a meridian-longitude pair of J_2 in S^3, and express $[\alpha] = p\mu_{J_2} + q\lambda_{J_2} \in H_1(\partial W_2)$.

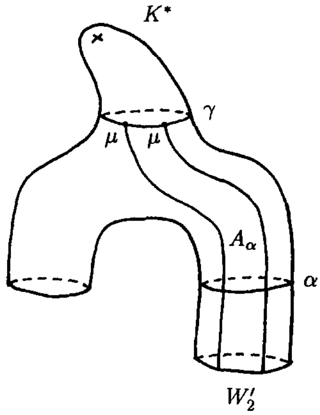


Fig. 10.

CLAIM 6.12. *Suppose that $p, q \geqslant 2$. Then $V'(J_2;\alpha)$ is reducible and K^* has non-zero winding number in V_α; in particular, K^* does not lie in a 3-ball in V_α.*

Proof. From the assumption we see that L_α is the lens space $L(p,q) \not\cong S^3$, hence the first assertion follows.

Since K' is a trivial knot in S^3 and K^* is its dual in $(K';\gamma) \cong S^2 \times S^1$, K^* intersects $\hat{S} = S^2 \times \{t\}$ ($t \in S^1$) algebraically once. Hence the winding number of K^* in the solid torus $V = (K';\gamma) - \text{int}\, W_1 = S^2 \times S^1 - \text{int}\, W_1$ is equal to one. Claim 6.10 shows that J_2 is a core of V. Then $(\mu_{J_2}, \lambda_{J_2})$ forms a basis of $H_1(V - \text{int}\, W_2) \cong Z \oplus Z$. Since $\text{wind}_V(K^*) = 1$, K^* is homologous to $x\mu_{J_2} + \lambda_{J_2}$ in $V - \text{int}\, W_2$ for some integer x. (K^* is suitably oriented.) Since $V_\alpha \cong V(J_2;\alpha)$, if $[K^*] = 0 \in H_1(V_\alpha)$, then $x\mu_{J_2} + \lambda_{J_2}$ is a multiple of $p\mu_{J_2} + q\lambda_{J_2}$. This implies that $q = \pm 1$, which contradicts our assumption. □

Then, as observed above, K^* is cabled in V_α for infinitely many slopes α by [34]. For such an α, let A_α be the cabling annulus for $K^* \subset V_\alpha$, so that the slope of a component of ∂A_α is $\mu(\subset \partial N(K^*))$ (Fig. 10).

CLAIM 6.13. *We cannot isotope A_α so that $A_\alpha \cap W_2' = \emptyset$.*

Proof. if we can isotope A_α off W_2', then A_α is contained in P. Since a hyperbolic manifold P cannot contain an essential annulus, this is impossible. □

We isotope A_α so that it intersects W_2' in meridian disks and $|A_\alpha \cap W_2'|$ is minimum. Then $A_\alpha - \text{int}\, W_2'$ is an incompressible and boundary-incompressible surface in P such that each of its boundary components on $\partial W_2'$ has the slope α.

Let E_i ($i = 1, 2$) be the exterior of a non-trivial knot, e.g., the exterior of the trefoil knot. Now we attach E_1 to P along $\partial N(K^*)$ via an identifying homeomorphism which sends a longitude of E_1 to μ on $\partial N(K^*)$, and attach E_2 to P along ∂V_α arbitrarily. We denote a resulting manifold by M; ∂M is the torus $\partial W_2'$. Notice that the construction of $M = P \cup E_1 \cup E_2$ does not depend on the choice of α. Then take disjoint, incompressible Seifert surfaces S and S' in E_1 such that $\partial S \cup \partial S' = \partial A_\alpha$. We can obtain a surface $F_\alpha = (A_\alpha - \text{int}\, W_2') \cup S \cup S'$ properly embedded in M such that each component of ∂F_α has the slope α. A standard argument shows that F_α is incompressible and boundary-incompressible in M. Therefore there are infinitely many slopes α on ∂M each of which can be realized by a boundary slope (i.e., a boundary of incompressible, boundary-incompressible surface). This contradicts the Hatcher's finiteness theorem for boundary slopes [35]. The proof of Proposition 6.8 is now completed. □

7. PROOF OF MAIN RESULTS

We are now in a position to prove Theorem 1.2.

THEOREM 1.2 (Non-simple Seifert fibred manifolds). *Let K be a non-hyperbolic knot in S^3. If $(K;r)$ is a non-simple Seifert fibred manifold, then one of the following holds:*

(1) *K is the trefoil knot, and $r = 0$.*

(2) *K is the $(2pq \pm 1, 2)$-cable of a (p,q)-torus knot, and $r = 4pq$.*

(3) *K has a companion solid torus V whose core is a torus knot and $V - K$ admits a complete hyperbolic structure in its interior, and r is an integer. Moreover, if both $(K;m_1)$ and $(K;m_2)$ are non-simple Seifert fibred manifolds, then $|m_1 - m_2| \leqslant 1$.*

(4) *K is the connected sum of two torus knots $T_{p_1,q_1} \# T_{p_2,q_2}$, and $r = p_1q_1 + p_2q_2$.*

Proof. Any surgeries on a trivial knot cannot produce non-simple manifolds. Hence our knot K is a non-trivial torus knot or a satellite knot.

If K is a non-trivial torus knot, then K is the trefoil knot and $r = 0$ (conclusion (1) above) by Proposition 2.1.

If K is a satellite knot, then by Corollary 6.3, K is of either type $(H - T^s)$ $(s \geqslant 1)$, $(Ca - T)$, or $(Co - T^s)$ $(s \geqslant 2)$.

Case 1. K is of type $(H - T^s)$ $(s \geqslant 1)$.

Proposition 6.8 implies $s = 1$, thus K has the form described in conclusion (3) of the theorem. The assertion about surgery slopes in (3) is proved in Proposition 5.5.

Case 2. K is of type $(Ca - T)$.

In this case, by Proposition 5.6 conclusion (2) of the theorem holds.

Case 3. K is of type $(Co - T^s)$ $(s \geqslant 2)$.

In this case, Proposition 6.6 gives conclusion (4) above.

Theorem 1.2 is thus proved. □

COROLLARY 1.5. (Seifert homology 3-spheres). *Let K be a satellite knot in S^3. If a non-trivial surgery $(K;r)$ is a Seifert homology 3-sphere, then $r = \pm 1$. Moreover, $(K;1)$ and $(K;-1)$ cannot both be Seifert fibred.*

Proof. Express $r = 1/n$. We divide into two cases whether K is cabled or not. First, we assume that K is a (p,q)-cable of, say k $(q \geqslant 2)$. If $(K;1/n)$ is a non-simple Seifert fibred manifold, then Theorem 1.2 implies $1/n = 4pq$, absurd. Hence, $(K;1/n)$ is a simple Seifert fibred manifold. Let V be tubular neighbourhood of k containing K in its interior. The argument in the proof of Theorem 1.4 shows that $V(K;1/n)$ is a solid torus. Since $K \subset V$ is a (p,q)-cable, it follows $|pqn - 1| = 1$ [3], so that $(p,q,n) = (1,2,1)$ or $(-1,2,-1)$.

Let us assume that K is a non-cable satellite knot in S^3. Suppose that $(K;1/n)$ is a simple Seifert fibred manifold. By the proof of Theorem 1.4, K has a companion solid torus V such that K is a 1-bridge braid in V and $V(K;1/n) \cong S^1 \times D^2$. By performing m meridional twists along V for some integer m, we get a new knot K_m as the image of K such that K_m satisfies

Convention 2.1 of [25] with respect to a meridian-longitude pair for V. Put $w = \mathrm{wind}_V(K) = \mathrm{wind}_V(K_m)$. Then [25, Lemma 3.2] implies that only $\pm(wt + d)$-surgery on K_m in V yields $S^1 \times D^2$ for some $d \in \{b, b+1\}$, where t $(1 \leqslant t \leqslant w-2)$ and b $(1 \leqslant b \leqslant w-2)$ are the twist number and the bridge width, respectively. The sign $\pm$ depends on the orientation convention. It follows that only $-mw^2 \pm (wt + d)$-surgery on K in V yields $S^1 \times D^2$, hence $1/n = -mw^2 \pm (wt + d)$. Since $3 \leqslant w$ (otherwise K is cabled) and $wt + d \leqslant w(w-2) + (w-1) = w^2 - w - 1$, a simple computation shows that $|-mw^2 \pm (wt+d)| \geqslant 4$. Thus, $(K; 1/n)$ cannot be a simple Seifert fibred manifold, and hence is a non-simple Seifert fibred manifold. The required result follows from Corollary 1.3. □

8. EXAMPLES–GRAPH KNOTS

A graph knot is a knot whose exterior is a graph manifold; i.e., each piece of the torus decomposition of the knot exterior is Seifert fibred. In this section we exhibit the examples of graph knots producing non-simple Seifert fibred manifolds by Dehn surgery. So conclusions (1), (2), and (4) of Theorem 1.2 are not vacant.

Example 8.1. Let K be the trefoil knot. Then $(K; 0)$ is a non-simple Seifert fibred manifold which is a torus bundle over S^1. This corresponds to (1) in Theorem 1.2.

Example 8.2. Let K be the $(2pq \pm 1, 2)$-cable of a (p, q)-torus knot. Then $(K; 4pq)$ is a non-simple Seifert fibred manifold over the projective plane with two exceptional fibres of indices $|p|$, q. This example corresponds to (2) in Theorem 1.2.

Remark. It is known that $(K; 4pq \pm 1) \cong L(4pq \pm 1, 4q^2)$ for the cable knots K in Example 8.2. In fact, these are all non-trivial surgeries on satellite knots which produce manifolds with cyclic fundamental groups [9, 10].

We now prove Example 8.2. Let C be the $(\pm 1, 2)$-cable knot in the standard solid torus V in S^3, and J a core of the complementary solid torus $S^3 - \mathrm{int}\, V$. Let $L \subset \partial V$ be a longitude of V. We first show that $V(C; 0) = (C; 0) - \mathrm{int}\, N(J)$ is a circle bundle over the Möbius band with $L(\subset \partial V(C; 0))$ a fibre. An ambient isotopy of S^3 exchanges J and C, thus J is the $(\pm 1, 2)$-cable in $S^3 - \mathrm{int}\, N(C)$ (Fig. 11).

We may assume that in $(C; 0) = S^2 \times S^1$, J intersects $S^2 \times \{t\}$ in two points for each $t \in S^1$. Hence, $V(C; 0) = (C; 0) - \mathrm{int}\, N(J)$ is an annulus bundle over S^1 in which fibres are the annuli $A_t = S^2 \times \{t\} - \mathrm{int}\, N(J)$, $t \in S^1$. Since each annulus A_t is foliated by circles parallel to a meridian of $N(J)$, $V(C; 0)$ is a circle bundle with L a fibre. As the annulus bundle over S^1, the monodromy of $V(C; 0)$ is the π-rotation of an annulus described in Fig. 12. We thus see that the base space of $V(C; 0)$ as the circle bundle is the Möbius band. (The base space can be lifted to $V(C; 0)$ as the obvious Möbius band spanned by J in Fig. 11.)

Now embed V in S^3 with the preferred framing such that the core of V is a (p, q)-torus knot; then C is the $(\pm 1, 2)$-cable of the (p, q)-torus knot. Let (μ, λ) be a meridian-longitude pair of V. We know that $V(C; 0)$ is a circle bundle over the Möbius band with a fibre on $\partial V(C; 0)$ representing $\lambda \in H_1(\partial V)$. On the other hand, a regular fibre $(\subset \partial V)$ of the Seifert fibration of $E(T_{p,q}) = S^3 - \mathrm{int}\, V$ represents $pq\mu + \lambda \in H_1(\partial V)$. To make the fibration of $V(C; 0)$ and $E(T_{p,q})$ match on the boundary, apply pq meridional twists along the knotted solid torus V. Then C becomes the $(2pq \pm 1, 2)$-cable of the (p, q)-torus knot, and a longitude

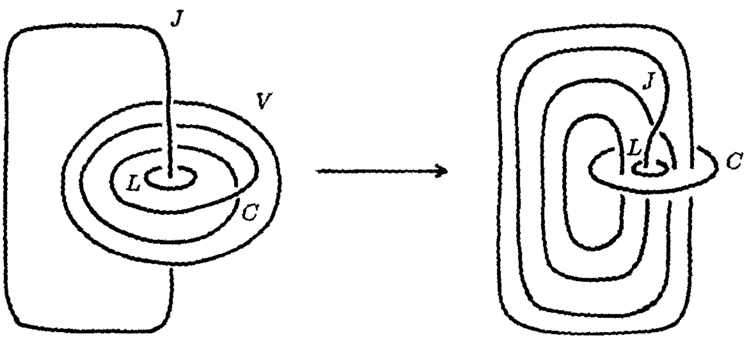


Fig. 11.

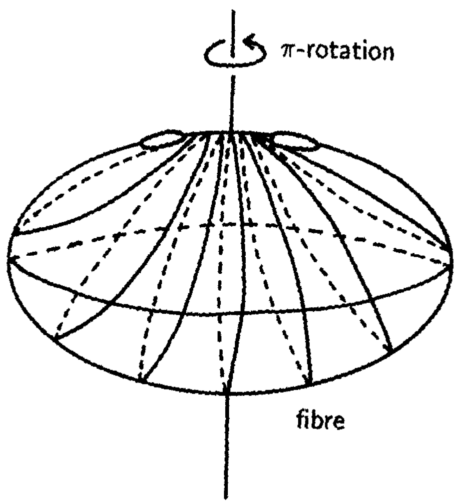


Fig. 12.

of V maps to a loop representing $pq\mu + \lambda$. Also note that a longitude of C maps to a loop representing $4pq$(meridian) + (longitude) $\in H_1(\partial N(C))$ via the twisting of V. Therefore, $(C; 4pq) = V(C; 4pq) \cup (S^3 - \operatorname{int} V)$ is Seifert fibred as desired. □

Example 8.3 (Ue [4]; Kalliongis and Tsau [5]). Let K be the connected sum of two torus knots T_{p_1,q_1}, T_{p_2,q_2}. Then $(T_{p_1,q_1} \# T_{p_2,q_2}; p_1q_1 + p_2q_2)$ is a non-simple Seifert fibred manifold over S^2 with four exceptional fibres of indices $|p_1|$, q_1, $|p_2|$, q_2. This corresponds to (4) in Theorem 1.2.

9. EXAMPLES – NON-GRAPH KNOTS $K_{p,q}(r,s)$

The purpose of this section is to present a new family of satellite knots of type $(H - T^1)$ producing non-simple Seifert fibred manifolds by surgery. To do that, we first construct knots in solid tori producing Seifert fibred manifolds with suitable properties.

Let V_1 be a standardly embedded solid torus in S^3 and V_2 the complementary solid torus $S^3 - \operatorname{int} V_1$. Let A be an annulus on ∂V_1 which winds around p times meridionally and q times longitudinally ($q > |p| \geq 2$), and set $A' = \partial V_1 - \operatorname{int} A$. Now we take a trivial knot

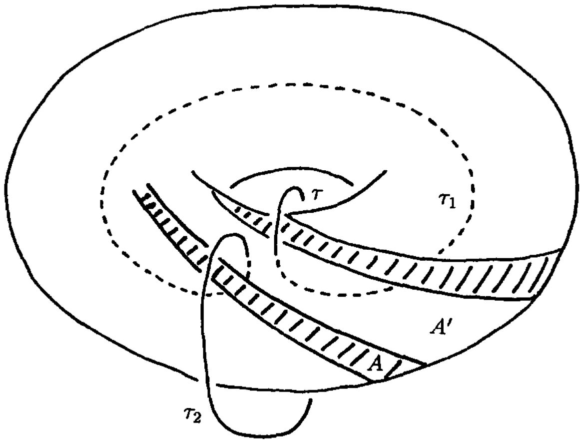


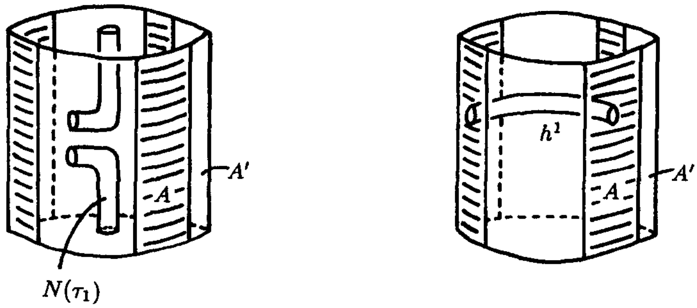
Fig. 13.

τ in S^3 depicted in Fig. 13, and put $\tau_i = \tau \cap V_i$ for $i = 1, 2$. Take a tubular neighbourhood $N(\tau)$ of τ such that $N(\tau) \cap A = \emptyset$. Let $V = S^3 - \operatorname{int} N(\tau)$, an unknotted solid torus. Then the core curve $C_{p,q}$ of A is a knot in V. It should be noted that a meridian of $N(\tau)$ is a longitude of V and $\operatorname{wind}_V(C_{p,q}) = \ell k(\tau, C_{p,q}) = p + q$. Furthermore, we can observe that the minimal geometric intersection number of $C_{p,q}$ with a meridian disk of V also equals $p + q$.

LEMMA 9.1. *The surgered manifold $V(C_{p,q}; pq)$ is a Seifert fibred manifold over the disk with two exceptional fibres of indices $|p|$ and q. Furthermore, a longitude of V is a regular fibre of $V(C_{p,q}; pq)$.*

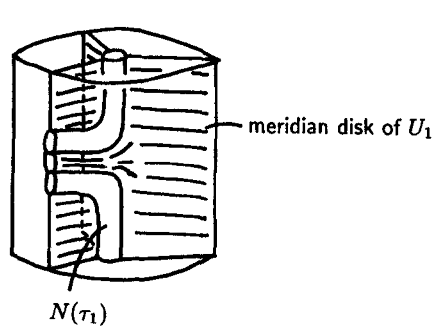
Proof. Choose a tubular neighbourhood $N(C_{p,q})$ so that $N(C_{p,q}) \cap \partial V_1 = A$. We identify $V(C_{p,q}; pq)$ with the union of two manifolds, which turn out to be solid tori. First note that $V - \operatorname{int} N(C_{p,q}) \cong (V_1 - \operatorname{int} N(\tau_1)) \cup (V_2 - \operatorname{int} N(\tau_2)) = M$, say, where $V_i - \operatorname{int} N(\tau_i)$ are pasted along $A' - \operatorname{int} N(\tau)$, an annulus with two holes; then the component of ∂M corresponding to $\partial N(C_{p,q})$ is the union of two copies of A. Since a component of $\partial A (\subset \partial N(C_{p,q}))$ has the slope pq in terms of a meridian-longitude pair of $C_{p,q}$, in $V(C_{p,q}; pq) = (V - \operatorname{int} N(C_{p,q})) \cup (S^1 \times D^2)$ the components of ∂A bound two disjoint, meridian disks of the glued solid torus $S^1 \times D^2$. The disks decompose $S^1 \times D^2$ into two 3-balls h_i^2, $i = 1, 2$, each of which is attached to $V_i - \operatorname{int} N(\tau_i)$ along a copy of A as a 2-handle. Hence, we can regard $V(C_{p,q}; pq)$ as the natural union of two manifolds $U_1 = (V_1 - \operatorname{int} N(\tau_1)) \cup_A h_1^2$ and $U_2 = (V_2 - \operatorname{int} N(\tau_2)) \cup_A h_2^2$. Since each τ_i is an unknotted arc in V_i, $V_i - \operatorname{int} N(\tau_i)$ is a handlebody of genus two. Furthermore, U_i is a solid torus. This can be explained for U_1 as follows. First "expanding" $N(\tau_1 \cup A')$ by an ambient isotopy of V_1, we can see $V_1 - \operatorname{int} N(\tau_1 \cup A') \cong N(A) \cup h^1$, where h^1 is a 1-handle depicted in Fig. 14. Clearly $V_1 - \operatorname{int} N(\tau) \cong V_1 - \operatorname{int} N(\tau_1 \cup A')$, hence U_1 is homeomorphic to $N(A) \cup_A h_1^2 \cup h^1$, a solid torus. A meridian disk of U_1 can be viewed as in Fig. 15. We remark that a meridian of $N(\tau_i)$ intersects that of U_1 algebraically q times ($q \geqslant 2$).

Let us see how the solid tori U_1 and U_2 are glued together. Set $T = \partial N(\tau_1) - \partial V_1$. Then T is an annulus on ∂U_1, and $U_1 \cap U_2$ is the complementary annulus $\partial U_1 - \operatorname{int} T$. Since a meridian of $N(\tau_1)$ intersects that of U_1 q times, the annulus T winds ∂U_1 around q times longitudinally. Hence the attaching annulus $U_1 \cap U_2$ also winds ∂U_1 around q times longitudinally. The same argument works for U_2, and shows that $U_1 \cap U_2$ winds



$q = 3$

Fig. 14.



$q = 3$

Fig. 15.

around ∂U_2 p times longitudinally. Therefore, $V(C_{p,q}; pq) \cong U_1 \cup U_2$ is a Seifert fibred manifold over the disk with two exceptional fibres of indices $|p|$ and q. From the construction, a longitude of V ($=$ a meridian of $N(\tau)$) is a regular fibre of $V(C_{p,q}; pq)$. $\square$

Now we construct a satellite knot producing a non-simple Seifert fibred manifold by Dehn surgery. Let $C_{p,q}$ and V be the knot and the unknotted solid torus constructed above. In the following, we assume that $\mathrm{wind}_V(C_{p,q}) = p + q \geqslant 2$. Let (m_V, ℓ_V) be a meridian-longitude pair of V. Choose $f\colon V \hookrightarrow S^3$ to be an orientation preserving embedding such that f sends a core of V to an (r,s)-torus knot, and $f(\ell_V) = \ell + rsm \in H_1(f(\partial V))$, where (m, ℓ) denotes a meridian-longitude pair of $f(V)$. Then set $K_{p,q}(r,s) = f(C_{p,q}) \subset S^3$; $K_{p,q}(r,s)$ is a satellite knot with a companion solid torus $f(V)$. We note here that since a torus knot is invertible, $K_{p,q}(r,s)$ is well-defined. That is, the choice of orientations of ℓ_V and ℓ does not affect the knot type of $K_{p,q}(r,s)$. Let $\gamma \subset \partial N(C_{p,q})$ be the slope corresponding to pq in terms of a meridian-longitude pair of $C_{p,q}$. ($f(\gamma)$ is a slope on $\partial N(K_{p,q}(r,s))$.) Then the $f(\gamma)$-surgery of $f(V)$ on $K_{p,q}(r,s)$ is a Seifert fibred manifold with $f(\ell_V)$ a regular fibre on the boundary, hence $(K_{p,q}(r,s); f(\gamma))$ is Seifert fibred. We need to parametrize the slope $f(\gamma)$ in terms of a meridian-longitude pair of $K_{p,q}(r,s)$. Since $\mathrm{wind}_{f(V)}(K_{p,q}(r,s)) = \mathrm{wind}_V(C_{p,q}) = p + q$ and f gives rs meridional twists, $f(\gamma)$ corresponds to $rs(p+q)^2 + pq$. It follows that $(K_{p,q}(r,s); rs(p+q)^2 + pq)$ is a non-simple Seifert fibred manifold over S^2 with four exceptional fibres of indices $|p|$, q, $|r|$, s.

CLAIM 9.2. *$V - C_{p,q}$ admits a complete hyperbolic structure in its interior.*

Proof. Since $\mathrm{wind}_V(C_{p,q}) \geqslant 2$, referring to the list of Theorem 1.2, we see that $K = K_{p,q}(r,s)$ is either the $(2rs \pm 1, 2)$-cable of an (r,s)-torus knot or a satellite knot of type $(H - T^1)$. In the first case, $(K; rs(p+q)^2 + pq)$ also admits a Seifert fibration whose base space is the projective plane by Example 8.2. This contradicts the uniqueness of Seifert fibrations of such a manifold [28]. Hence K is a satellite knot of type $(H - T^1)$. □

By the construction, we can take any coprime integers p, q independent of r, s. We thus obtain:

PROPOSITION 9.3. *For any non-trivial torus knot $T_{r,s}$ and any coprime integers p, q ($q > |p| \geqslant 2$ and $p + q \geqslant 2$), there is a satellite knot $K_{p,q}(r,s)$ satisfying the following properties:*

(1) *$K_{p,q}(r,s)$ has a companion solid torus V such that* $\mathrm{wind}_V(K_{p,q}(r,s)) = p + q$ *and a core of V is $T_{r,s}$.*

(2) *$V - K_{p,q}(r,s)$ admits a complete hyperbolic structure in its interior.*

(3) *$(K_{p,q}(r,s); rs(p+q)^2 + pq)$ is a Seifert fibred manifold over S^2 with four exceptional fibres of indices $|p|, q, |r|, s$.*

Remark. If $p + q = 1$, then $K_{p,q}(r,s)$ is the connected sum of two torus knots $T_{p,q}$ and $T_{r,s}$.

PROPOSITION 9.4 (Classification of $K_{p,q}(r,s)$). *$K_{p_1,q_1}(r_1,s_1)$ and $K_{p_2,q_2}(r_2,s_2)$ ($q_i > |p_i| \geqslant 2$, $p_i + q_i \geqslant 2$ and $s_i > |r_i| \geqslant 2$) are ambient isotopic in S^3 if and only if $(p_1,q_1,r_1,s_1) = (p_2,q_2,r_2,s_2)$.*

Proof. Assume that $K_{p_1,q_1}(r_1,s_1)$ and $K_{p_2,q_2}(r_2,s_2)$ are isotopic in S^3. By the proposition above, $K_{p_1,q_1}(r_1,s_1)$ produces non-simple Seifert fibred manifolds by $r_is_i(p_i+q_i)^2 + p_iq_i$-surgeries for both $i = 1,2$. Hence by Corollary 1.3, we have $|(r_1s_1(p_1+q_1)^2 + p_1q_1) - (r_2s_2(p_2+q_2)^2 + p_2q_2)| \leqslant 1$. On the other hand, by the uniqueness of the torus decomposition, we have $(r_1,s_1) = (r_2,s_2)$. In addition, since $p_i + q_i$ equals the winding number of $K_{p_i,q_i}(r_i,s_i)$ in a companion solid torus whose core is T_{r_i,s_i}, we see that $p_1 + q_1 = p_2 + q_2$. So the above inequality implies that $p_1q_1 = p_2q_2$ or $p_1q_1 = p_2q_2 \pm 1$. In the first case, clearly $(p_1,q_1) = (p_2,q_2)$. We exclude the latter possibility as follows. Since p_1 and q_1 are coprime integers, $p_1 + q_1$ and p_1q_1 have different parity. Hence, $p_2 + q_2 (= p_1 + q_1)$ and $p_2q_2 (= p_1q_1 \pm 1)$ have the same parity. This implies that both p_2 and q_2 are even integers, a contradiction. Thus we obtain the required result. □

Example 9.5. $(K_{-11,13}(4,9); 1)$ is a non-simple Seifert homology 3-sphere over S^2 with four exceptional fibres of indices 11, 13, 4, 9. This homology 3-sphere cannot be obtained by surgery on any graph knots.

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